JOURNAL OF APPROXIMATION THEORY **86**, 229–239 (1996)
ARTICLE NO. 0064

The Generalized Markov–Stieltjes Inequality for Birkhoff Quadrature Formulas*

Ying Guang Shi

Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, People's Republic of China

Communicated by Borislav Bojanov

Received August 15, 1994; accepted in revised form September 4, 1995

The generalized Markov–Stieltjes inequalities for several kinds of generalized Gaussian Birkhoff quadrature formulas are given. © 1996 Academic Press, Inc.

1

This paper deals with the generalized Markov–Stieltjes inequality for a generalized Gaussian Birkhoff quadrature formula (GGBQF). The generalized Markov–Stieltjes inequality for a Gaussian quadrature formula has extensive applications in the covergence of product integration rules and an estimation of the rate of convergence of a Gaussian quadrature formula for singular integrands. For full information on this subject see the introduction and the references in [3].

Let $\alpha(x)$ be a nondecreasing continuous function on $[a, b]$. Then there exists a unique set of nodes

$$X\colon a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b \tag{1.1}$$

and a Gauss quadrature formula

$$\int_a^b f(x)\, d\alpha(x) = \sum_{i=1}^{m} \lambda_i f(x_i), \tag{1.2}$$

determined uniquely from being exact for all $f \in \mathbf{P}_{2m-1}$, the space of all polynomials of degree at most $2m-1$. As we know, for this Gauss

* This subject was supported by the National Natural Science Foundation of China.

0021-9045/96 $18.00

quadrature formula with simple nodes the Markof–Stieltjes inequality holds [5, p. 49]:

$$\sum_{i=1}^{r-1} \lambda_i \leqslant \int_a^{x_r} d\alpha(x) \leqslant \sum_{i=1}^{r} \lambda_i, \qquad 1 \leqslant r \leqslant m. \tag{1.3}$$

Recently, the generalized Markov–Stieltjes inequality was extended to Turán quadrature formulas with nodes of odd multiplicities by Gori [3]. To state her results let μ_i, $i=1, ..., m$, be odd integers and $n = m-1+\sum_{i=1}^{m} \mu_i$. Then there exists a unique set of nodes X such that the Gaussian quadrature formula

$$\int_a^b f(x)\, d\alpha(x) = \sum_{i=1}^{m} \sum_{j=0}^{\mu_i - 1} a_{ij} f^{(j)}(x_i) \tag{1.4}$$

is exact for all $f \in \mathbf{P}_n$. For this Gaussian quadrature formula the generalized Markov–Stieltjes inequality holds [3]:

THEOREM A. *Given an r, $1 \leqslant r \leqslant m$, if f is n-absolutely monotone in $(a, x_r]$, i.e.,*

$$f^{(k)}(x) \geqslant 0, \qquad x \in (a, x_r], \qquad k = 0, 1, ..., n, \tag{1.5}$$

then

$$\sum_{i=1}^{r-1} \sum_{j=0}^{\mu_i - 1} a_{ij} f^{(j)}(x_i) \leqslant \int_a^{x_r} f(x)\, d\alpha(x) \leqslant \sum_{i=1}^{r} \sum_{j=0}^{\mu_i - 1} a_{ij} f^{(j)}(x_i). \tag{1.6}$$

On the other hand, we have known of the existence of a generalized Gaussian Hermite quadrature formula (GGHQF) with multiple nodes and even of a GGBQF with Birkhoff nodes. It is natural to ask if the generalized Markov–Stieltjes inequality can be extended to these quadrature formulas. The aim of this paper is to answer this question. Using a modification of the idea in [3] we shall prove that the generalized Markov–Stieltjes inequality (1.6) can be extended to a GGHQF (Section 2), but cannot be extended to a GGBQF (Section 3); the Markov–Stieltjes inequality (1.3) can be extended to two kinds of GGBQFs (Sections 4 and 5), although at present we do not know whether these formulas are unique.

2

In what follows we shall use the definitions and notations of [4]. Let $E = (e_{ij})_{i=0,\, j=0}^{m+1\ n}$ be an incidence matrix with entries consisting of zeros and ones and satisfying $|E| := \sum_{i,j} e_{ij} < n+1$ (here we allow a zero row). For each i, $0 \leqslant i \leqslant m+1$, let μ_i denote the smallest index j such that $e_{ij} = 0$. Then the following GGHQF holds [1]:

THEOREM B. *Let an* $(m+2) \times (n+1)$ *incidence matrix* E *satsify*

$$\mu_0 \geqslant 0, \qquad \mu_{m+1} \geqslant 0, \qquad \mu_i > 0, \qquad i = 1, ..., m$$

and

$$|E| = \sum_{i=0}^{m+1} \mu_i = n + 1 - m.$$

Then there exists a unique set of nodes X *such that the quadrature formula*

$$\int_a^b \sigma(X; x)\, f(x)\, d\alpha(x) = \sum_{e_{ij}=1} a_{ij} f^{(j)}(x_i) \tag{2.1}$$

is exact for all $f \in \mathbf{P}_n$, *where*

$$\sigma(x) := \sigma(X; x) := \operatorname{sgn} \prod_{i=1}^{m} (x - x_i)^{v_i} \tag{2.2}$$

and

$$v_0 = \mu_0, \qquad v_{m+1} = \mu_{m+1}, \qquad v_i = \mu_i + 1, \qquad i = 1, ..., m. \tag{2.3}$$

Here we quote a fundamental lemma [2, p. 30] needed later.

LEMMA A. *Given a* $\xi \in (a, b)$ *let* f *satisfy*

$$f^{(k)}(x) > 0, \qquad x \in (a, \xi], \qquad k = 0, 1, ..., n \tag{2.4}$$

and let $P \in \mathbf{P}_{n-1}$. *Then the number of zeros (counting multiplicities) of the function*

$$F_\xi(x) = \begin{cases} f(x) - P(x), & x \in [a, \xi) \\ P(x), & x \in [\xi, b] \end{cases} \tag{2.5}$$

is not greater than n.

Remark. In what follows we agree that for $1 \leqslant k \leqslant n$

$$F_\xi^{(k)}(x) = \begin{cases} f^{(k)}(x) - P^{(k)}(x), & x \in [a, \xi) \\ P^{(k)}(x), & x \in [\xi, b]. \end{cases}$$

Now we can state the main result in this section.

THEOREM 1. *Let the assumptions of Theorem B be satified. Given an r, $1 \leqslant r \leqslant m$, if f satisfies (1.5) then*

$$(-1)^{\sum_{i=r}^{m} \nu_i} \int_a^{x_r} \sigma(X; x)\, f(x)\, d\alpha(x) \geqslant (-1)^{\sum_{i=r}^{m} \nu_i} \sum_{e_{ij}=1,\, i<r} a_{ij} f^{(j)}(x_i), \tag{2.6}$$

$$(-1)^{\sum_{i=r+1}^{m} \nu_i} \int_a^{x_r} \sigma(X; x)\, f(x)\, d\alpha(x) \leqslant (-1)^{\sum_{i=r+1}^{m} \nu_i} \sum_{e_{ij}=1,\, i<r+1} a_{ij} f^{(j)}(x_i). \tag{2.7}$$

Here ν_i's are defined by (2.3)

Proof. Let us prove (2.6).

If $r=1$ and $\nu_0=0$ then the right side of (2.6) is zero. Meanwhile, since

$$\sigma(x) = (-1)^{\sum_{i=1}^{m} \nu_i}$$

for $x \in (a, x_1)$, the left side of (2.6) is $\int_a^{x_1} f(x)\, d\alpha(x) \geqslant 0$. Thus in this case (2.6) holds.

Now let $r>1$ or $\nu_0>0$.

First, we are going to show that (2.6) holds if f satisfies (2.4) with $\xi = x_r$.

Let $\xi \in (x_{r-1}, x_r)$ be arbitrary. Choose $P \in \mathbf{P}_{n-1}$ so that $F_\xi(x)$ defined in (2.5) is annihilated by the pair (E', X), where E' is obtained from E by adding a 1 to position (i, μ_i), $i = 1, ..., r-1, r+1, ..., m$.

We claim that

$$\Phi(x) = [\operatorname{sgn}(x - x_r)]\, \sigma(x)\, F_\xi(x) \tag{2.8}$$

does not change sign in (a, b).

In fact, suppose to the contrary that $\Phi(x)$ changes sign at $z \in (a, b)$. If $z \notin X$ then $F_\xi(z)=0$. If $z = x_i$, $1 \leqslant i \leqslant m$, then $\Phi(x_i - \delta)\, \Phi(x_i + \delta) < 0$ holds for all small $\delta > 0$, which implies

$$(-1)^{\nu_i'} F_\xi(x_i - \delta)\, F_\xi(x_i + \delta) < 0,$$

where $\nu_i' = \nu_i$ for $i \neq r$ and $\nu_i' = \nu_i - 1$ for $i = r$. By definition we conclude $F_\xi^{(\nu_i')}(x_i) = 0$. That is to say, the number of zeros of F_ξ is not less than $|E'| + 1 = n+1$. This contradicts Lemma A. This proves that $\Phi(x)$ does not change sign in (a, b).

Since $\xi \in (x_{r-1}, x_r)$ is arbitrary, $P(x)$ does not change sign in (x_{r-1}, x_r). Meanwhile $P(x_{r-1}) = f(x_{r-1}) > 0$. So $P(x) > 0$, $x \in (x_{r-1}, x_r)$. This means

$$(-1)^{\sum_{i=r}^{m} v_i} \Phi(x) \leqslant 0, \qquad x \in [a, b]. \tag{2.9}$$

Thus

$$-\int_a^{x_r} (-1)^{\sum_{i=r}^{m} v_i} \Phi(x)\, d\alpha(x) \geqslant 0 \geqslant \int_{x_r}^{b} (-1)^{\sum_{i=r}^{m} v_i} \Phi(x)\, d\alpha(x). \tag{2.10}$$

Hence as $\xi \to x_r$ we obtain

$$\begin{aligned}(-1)^{\sum_{i=r}^{m} v_i} \int_a^{x_r} \sigma(x) f(x)\, d\alpha(x) &\geqslant (-1)^{\sum_{i=r}^{m} v_i} \int_a^{b} \sigma(x) P(x)\, d\alpha(x) \\ &= (-1)^{\sum_{i=r}^{m} v_i} \sum_{e_{ij}=1,\, i<r} a_{ij} f^{(j)}(x_i). \end{aligned} \tag{2.11}$$

Next, if f satsifies (1.5) only, then we consider $f_\varepsilon(x) = f(x) + \varepsilon e^x$, $\varepsilon > 0$. Applying the above conclusion and letting $\varepsilon \to 0$ yields (2.6) (see [3]).

To prove (2.7) we use the same arguments as those above except for $\xi \in (x_r, x_{r+1})$. In this case $\Phi(x)$ does not change sign in (a, b) and $P(x) > 0$, $x \in (x_r, x_{r+1})$. This means

$$(-1)^{\sum_{i=r+1}^{m} v_i} \Phi(x) \geqslant 0, \qquad x \in [a, b].$$

Thus

$$-\int_a^{x_r} (-1)^{\sum_{i=r+1}^{m} v_i} \Phi(x)\, d\alpha(x) \leqslant 0 \leqslant \int_{x_r}^{b} (-1)^{\sum_{i=r+1}^{m} v_i} \Phi(x)\, d\alpha(x),$$

or, equivalently

$$\begin{aligned}&(-1)^{\sum_{i=r+1}^{m} v_i} \int_a^{x_r} \sigma(x) f(x)\, d\alpha(x) \\ &\leqslant (-1)^{\sum_{i=r+1}^{m} v_i} \int_a^{b} \sigma(x) P(x)\, d\alpha(x) \\ &= (-1)^{\sum_{i=r+1}^{m} v_i} \sum_{e_{ij}=1,\, i<r+1} a_{ij} f^{(j)}(x_i). \quad \blacksquare \end{aligned}$$

3

The following example illustrates that a generalized Markov–Stieltjes inequality (1.6) can not be extended to a GGBQF.

It is easy to check that

$$\int_{-1}^{1} f(x)\,dx = f(-t) + f(t) + \tfrac{1}{135} f^{(4)}(0), \qquad t = 3^{-1/2}$$

is exact for every $f \in \mathbf{P}_5$. Meanwhile for $f(x) = (1+x)^7$, which satisfies (1.5) on $[-1, 1]$, we have

$$\int_{-1}^{t} f(x)\,dx = \tfrac{1}{8}(1+t)^8 \simeq 4.79,$$

$$\sum_{x_i < t} a_{ij} f^{(j)}(x_i) = f(-t) + \tfrac{1}{135} f^{(4)}(0) = (1-t)^7 + \tfrac{56}{9} > \tfrac{56}{9} > 4.79.$$

Thus (2.6) is violated.

4

In [6] we obtained the GGBQF as follows.

Theorem C. *Let an* $(m+2) \times (n+1)$ *incidence matrix* E *satsify the delayed Pólya conditions*

$$\begin{aligned} M_k(E) &:= \sum_{j=0}^{k} \sum_{i=0}^{m+1} e_{ij} \geqslant k+1-\rho, \qquad k = 0, 1, ..., n, \\ M_n(E) &= |E| = n+1-\rho \end{aligned} \tag{4.1}$$

with the constant $\rho = p$, $0 \leqslant p \leqslant m$, *and contain no odd non-Hermitian sequences in the interior rows* $1 \leqslant i \leqslant m$. *Then for any prescribed* p *interior rows* i_k, $k = 1, ..., p$, *there exists a set of nodes* (1.1) *such that* (2.1) *holds for all* $f \in \mathbf{P}_n$, *where*

$$v_i = \begin{cases} \mu_i + 1, & i = i_1, ..., i_p, \\ \mu_i, & \text{otherwise}. \end{cases} \tag{4.2}$$

If we put

$$\lambda_i = \begin{cases} a_{i0}, & e_{i0} = 1 \\ 0, & e_{i0} = 0 \end{cases} \tag{4.3}$$

then for this GGBQF the following is true.

THEOREM 2. *Under the assumptions of Theorem C we have*

$$(-1)^{\sum_{i=r}^{m} v_i} \int_a^{x_r} \sigma(X; x)\, d\alpha(x) \geqslant (-1)^{\sum_{i=r}^{m} v_i} \sum_{i=0}^{r-1} \lambda_i, \qquad 1 \leqslant r \leqslant m, \tag{4.4}$$

$$(-1)^{\sum_{i=r+1}^{m} v_i} \int_a^{x_r} \sigma(X; x)\, d\alpha(x) \leqslant (-1)^{\sum_{i=r+1}^{m} v_i} \sum_{i=0}^{r} \lambda_i, \qquad 1 \leqslant r \leqslant m. \tag{4.5}$$

Proof. Let us give the proof of (4.4) only, the one of (4.5) being similar. Now let E' be otained from E by adding a 1 to the position (i, μ_i) for $i = i_1, ..., i_p$ and then by dropping the 1 in the position $(r, v_r - 1)$. We distinguish two cases.

Case 1. $v_0 = v_1 = \cdots = v_{r-1} = 0$. In this case $\lambda_0 = \lambda_1 = \cdots = \lambda_{r-1} = 0$ by (4.3) and hence the right side of (4.4) is zero. On the other hand, since

$$\sigma(x) = (-1)^{\sum_{i=r}^{m} v_i}, \qquad x \in (a, x_r), \tag{4.6}$$

the left side of (4.4) equals $\int_a^{x_r} d\alpha(x) \geqslant 0$. Thus in this case (4.4) holds.

Case 2. $v_s > v_{s+1} = \cdots = v_{r-1} = 0$ with $0 \leqslant s \leqslant r-1$. In this case for $\xi \in (x_s, x_r)$ and $f = 1$ choose $P \in \mathbf{P}_{n-1}$ so that $F_\xi(x)$ in (2.5) is annihilated by the pair E', X.

We claim that $\Phi(x)$ in (2.8) does not change sign in (a, b).

In fact, suppose the contrary that $\Phi(x)$ changes sign at $z \in (a, b)$. If $z \notin X$ then $F_\xi(z) = 0$ and add a new Lagrangian row. If $z = x_i$, $1 \leqslant i \leqslant m$, then by the arguments used in the proof of Theorem 1 we conclude $F_\xi^{(v_i')}(x_i) = 0$, where v_i' is the smallest index j such that $e_{ij}' = 0$ in E', since E contains no odd non-Hermitian sequences in the interior rows. In this case we add a 1 to position (i, v_i'). Let $\bar{E}$ be obtained from E' by the above process. Then $F_\xi(x)$ is annihilated by the pair $\bar{E}$, $Z = X \cup \{z\}$, where $\bar{E}$ is still regular [4, p. 10]. Let $Z_1 = \{x \in Z : x < \xi\}$ and $Z_2 = Z \backslash Z_1$. Let $\bar{E}_1$ and $\bar{E}_2$ be the submatrices of $\bar{E}$ corresponding to Z_1 and Z_2. Then by the arguments used in the proof of Lemma 1.7 in [4, p. 10] we can get matrices $\bar{E}_1'$ and $\bar{E}_2'$ and sets Z_1' and Z_2' that annihilate the derivatives $f' - P'$ and P', respectively. Meanwhile $|\bar{E}_1'| \geqslant |\bar{E}_1|_{-1}$ and $|\bar{E}_2'| \geqslant |\bar{E}_2| - 1$. But $f' - P' = -P'$. So the pair $\bar{E}_1'$, Z_1' annihilates P', too. Hence the pair $\bar{E}' = \bar{E}_1' + \bar{E}_2'$, $Z_1' \cup Z_2'$, which is regular, annihilates the derivative P'. Noting that $P' \in \mathbf{P}_{n-2}$ and $|\bar{E}'| \geqslant |\bar{E}| - 2 \geqslant n-1$, we conclude $P' = 0$, i.e., $P = \text{const}$. This contradicts the definition of P and proves the claim.

Since $\xi \in (x_s, x_r)$ is arbitrary, $P(x)$ does not change sign in (x_s, x_r). Meanwhile $P(x_s) = 1$. So $P(x) > 0$, $x \in (x_s, x_r)$. Now we see that (2.9), (2.10), and (2.11) with $f = 1$ hold. This proves (4.4). ▮

5

For each i, $0 \leqslant i \leqslant m+1$, let the bottom sequence in the ith row of the matrix E be $e_{i, k_i} = \cdots = e_{i, k_i + \bar{\mu}_i - 1} = 1$ (i.e., $e_{ij} = 0$ for $j < k_i$ and $j = k_i + \bar{\mu}_i$). We agree that $\bar{\mu}_0 = 0$ and $k_0 = k_1$ if $e_{0j} \equiv 0$ and that $\bar{\mu}_{m+1} = 0$ and $k_{m+1} = k_m$ if $e_{m+1, j} \equiv 0$. Then we have [7]

THEOREM D. *Let an* $(m+2) \times (n+1)$ *incidence matrix* E *satisfy*

$$\begin{cases} k_{i+1} \leqslant k_i \leqslant k_{i+1} + \bar{\mu}_{i+1}, & i = 0, ..., I-1, \\ k_i = 0, & i = I, ..., J, \\ k_{i-1} \leqslant k_i \leqslant k_{i-1} + \bar{\mu}_{i-1}, & i = J+1, ..., m+1, \end{cases} \tag{5.1}$$

for some $0 \leqslant I \leqslant J \leqslant m+1$, *and contain no odd non-bottom sequences in the interior rows* $1 \leqslant i \leqslant m$. *Assume that the number of odd bottom sequences in the rows* $1 \leqslant i \leqslant I-1$ *and* $J+1 \leqslant i \leqslant m$ *is* q *and* E *satifies the delayed Pólya conditions* (4.1) *with the constant* $\rho = p$, $q \leqslant p \leqslant q + J - I + 1$. *Then for any prescribed* $p - q$ *interior rows* i_k, $I \leqslant i_1 < \cdots < i_{p-q} \leqslant J$, *there exists a set of nodes* (1.1) *with* $d\alpha(x) \equiv dx$ *such that* (2.1) *holds for all* $f \in \mathbf{P}_n$, *where*

$$v_i = \begin{cases} 0, & i = 1, ..., I-1, J+1, ..., m, \\ \bar{\mu}_i + 1, & i = i_1, ..., i_{p-q}, \\ \bar{\mu}_i, & \text{otherwise}. \end{cases} \tag{5.2}$$

Then for this GGBQF the following holds.

THEOREM 3. *Under the assumptions of Theorem D we have*

$$(-1)^{\sum_{i=r}^m v_i} \int_a^{x_r} \sigma(X; x)\, dx \geqslant (-1)^{\sum_{i=r}^m v_i} \sum_{i=0}^{r-1} \lambda_i, \qquad I \leqslant r \leqslant J, \tag{5.3}$$

$$(-1)^{\sum_{i=r+1}^m v_i} \int_a^{x_r} \sigma(X; x)\, dx \leqslant (-1)^{\sum_{i=r+1}^m v_i} \sum_{i=0}^{r} \lambda_i, \qquad I \leqslant r \leqslant J. \tag{5.4}$$

Here λ_i*'s are given by* (4.3)

Proof. Let us prove (5.3).

If $r=I$ then the right side of (5.3) is zero. On the other hand, since (4.6) is true, the left side of (5.3) equals $\int_a^{x_r} dx > 0$. Thus in this case (5.3) holds.

Now let $r>I$ and let $\xi \in (x_{r-1}, x_r)$ be arbitrary. Assume that the q rows containing odd bottom sequences are $i_{p-q+1}, ..., i_p \in \{1, ..., I-1, J+1, ..., m\}$. Put

$$\omega_i = \begin{cases} \bar{\mu}_i + 1, & i = i_1, ..., i_p, \\ \bar{\mu}_i, & \text{otherwise.} \end{cases}$$

It is easy to check that $\sigma(x) = \operatorname{sgn} \prod_{i=1}^m (x-x_i)^{\omega_i}$, where $\sigma(x)$ is defined by (2.2) and (5.2). Let E' be obtained from E by adding a 1 to the position $(i, k_i + \bar{\mu}_i)$ for $i = i_1, ..., i_p$ and then by dropping the 1 in position $(r, k_r + \omega_r - 1)$. In his case for $f=1$ choose $P \in \mathbf{P}_{n-1}$ so that $F_\xi(x)$ in (2.5) is annihilated by the pair E', X. Put

$$Q(x) := \begin{cases} (a-x)^{k_i} F_\xi^{(k_i)}(x)/(k_i)!, \\ \qquad x \in [x_i, x_{i+1}), \qquad i \leqslant I-1, \\ (b-x)^{k_{i+1}} F_\xi^{(k_{i+1})}(x)/(k_{i+1})!, \\ \qquad x \in [x_i, x_{i+1}), \qquad I \leqslant i \leqslant m-1, \\ (b-x)^{k_{m+1}} F_\xi^{(k_{m+1})}(x)/(k_{m+1})!, \\ \qquad x \in [x_m, b], \qquad i = m. \end{cases} \tag{5.5}$$

First, we are going to prove

$$\int_a^{x_r} \sigma(x)\, Q(x)\, dx = \int_a^{x_r} \sigma(x)\, F_\xi(x)\, dx, \tag{5.6}$$

$$\int_{x_r}^b \sigma(x)\, Q(x)\, dx = \int_{x_r}^b \sigma(x)\, F_\xi(x)\, dx. \tag{5.7}$$

Let us give the proof of (5.6) only, that of (5.7) being similar. We see

$$\begin{aligned} \int_a^{x_r} \sigma(x)\, Q(x)\, dx &= \sum_{i=0}^{I-1} \int_{x_i}^{x_{i+1}} \frac{\sigma(x)}{(k_i)!} (a-x)^{k_i} F_\xi^{(k_i)}(x)\, dx + \int_{x_I}^{x_r} \sigma(x)\, F_\xi(x)\, dx \\ &= \int_a^{x_r} \sigma(x)\, F_\xi(x)\, dx + \sigma(a) \sum_{i=0}^{I-1} \sum_{j=1}^{k_i} \frac{1}{j!} \\ &\quad \times [(a-x_{i+1})^j F_\xi^{(j-1)}(x_{i+1}) - (a-x_i)^j F_\xi^{(j-1)}(x_i)] \\ &= \int_a^{x_r} \sigma(x)\, F_\xi(x)\, dx + \sigma(a) \sum_{i=1}^{I} \sum_{j=k_i}^{k_{i-1}-1} \frac{(a-x_i)^{j+1} F_\xi^{(j)}(x_i)}{(j+1)!} \\ &= \int_a^{x_r} \sigma(x)\, F_\xi(x)\, dx. \end{aligned}$$

Next, we claim that $\Phi(x)=[\operatorname{sgn}(x-x_r)]\,\sigma(x)\,Q(x)$ does not change sign in (a,b).

In fact, suppose to the contrary that $\Phi(x)$ changes sign at $z\in(a,b)$. If $z\notin X$, say, $z\in(x_i,x_{i+1})$, $i\leqslant I-1$, then $F_\xi^{(k_i)}(z)=0$ and add a new row $(0,\ldots,0,1,0,\ldots,0)$, where the 1 lies on the k_i th column. If $z=x_i$, $1\leqslant i\leqslant m$, say, $i\leqslant I-1$, then $\Phi(x_i-\delta)\,\Phi(x_i+\delta)<0$ holds for all small $\delta>0$, which implies

$$[(-1)^{\omega_i+k_{i-1}}F_\xi^{(k_{i-1})}(x_i-\delta)][(-1)^{k_i}F_\xi^{(k_i)}(x_i+\delta)]\begin{cases}<0, & i\neq r\\ >0, & i=r.\end{cases}$$

Here we need a simple observation: If a polynomial $g(x)$ satisfies $g(y)=0$, then $g'(y-\delta)\,g(y-\delta)\leqslant 0$ and $g'(y+\delta)\,g(y+\delta)\geqslant 0$ hold for all small $\delta>0$. Thus remembering

$$F_\xi^{(k_i)}(x_i)=F_\xi^{(k_i+1)}(x_i)=\cdots=F_\xi^{(k_i+\omega_i-1)}(x_i)=0,\qquad i\neq r$$

and

$$F_\xi^{(k_i)}(x_i)=F_\xi^{(k_i+1)}(x_i)=\cdots=F_\xi^{(k_i+\omega_i-2)}(x_i)=0,\qquad i=r,$$

by induction we obtain

$$[(-1)^{\omega_i+k_{i-1}+k_i+\omega_i-k_{i-1}}F_\xi^{(k_i+\omega_i)}(x_i-\delta)]\\ \times[(-1)^{k_i}F_\xi^{(k_i+\omega_i)}(x_i+\delta)]\leqslant 0,\qquad i\neq r$$

and

$$[(-1)^{\omega_i+k_{i-1}+k_i+\omega_i-k_{i-1}-1}F_\xi^{(k_i+\omega_i-1)}(x_i-\delta)]\\ \times[(-1)^{k_i}F_\xi^{(k_i+\omega_i-1)}(x_i+\delta)]\geqslant 0,\qquad i=r,$$

or, equivalently,

$$F_\xi^{(k_i+\omega_i)}(x_i-\delta)\,F_\xi^{(k_i+\omega_i)}(x_i+\delta)\leqslant 0,\qquad i\neq r,$$

and

$$F_\xi^{(k_i+\omega_i-1)}(x_i-\delta)\,F_\xi^{(k_i+\omega_i-1)}(x_i+\delta)\leqslant 0,\qquad i=r,$$

hold for all small $\delta>0$. This explains shortly the reason for writing v' instead of $k_i+\omega_i$ or $k_i+\omega_i-1$, respectively. Hence we can obtain a new zero $F_\xi^{(v_i')}(x_i)=0$, where v_i' is the smallest index j such that $j\geqslant k_i$ and $e'_{ij}=0$ in E', since E contains no odd non-bottom sequences in the interior rows. In this case we add a 1 to the poition (i,v_i'). Let E^* be otained from E' by the above process. Then F_ξ is annihilated by the pair E^*, $Z=X\cup\{z\}$.

By the arguments used in the proof of Theorem 2 we can get a contradiction. This proves that Φ does not change sign.

Since $\xi \in (x_{r-1}, x_r)$ is arbitrary, $P(x)$ does not change sign in (x_{r-1}, x_r). Meanwhile $P(x_{r-1}) = 1$. So $P(x) > 0$, $x \in (x_{r-1}, x_r)$. Now we see that (2.9) and (2.10) hold. By (5.6) and (5.7) it follows from (2.10) that

$$\int_a^{x_r} (-1)^{\sum_{i=r}^m v_i} \sigma(x)\, F_\xi(x)\, dx(x) \geqslant 0 \geqslant \int_{x_r}^b (-1)^{\sum_{i=r}^m v_i} \sigma(x)\, F_\xi(x)\, d\alpha(x).$$

Hence (2.11) occurs. This proves (5.3).

We can prove (5.4) similarly. ∎

REFERENCES

1. B. D. Bojanov, D. Braess, and N. Dyn, Generalized Gaussian quadrature formulas, *J. Approx. Theory* **46** (1986), 335–353.
2. G. Frend, "Orthogonal Polynomials," Akad. Kiadó, Budapest/Pergamon, Oxford, 1971.
3. L. Gori, The Posse–Markov–Stieltjes inequality for Turán type quadrature sums, *in* "Progress in Approximation Theory" (P. Nevai and A. Pinkus, Eds.), pp. 453–466, Academic Press, New York, 1991.
4. G. G. Lorentz, K. Jetters, and S. D. Riemenschneider, "Birkhoff Interpolation," Addison–Wesley, Reading, MA/London, 1983.
5. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1939.
6. Y. G. Shi, Generalized Gaussian Birkhoff quadrature formulas, *J. Approx. Theory* **82** (1995), 392–398.
7. Y. G. Shi, Existence of generalized Gaussian Birkhoff quadrature formulas, *Acta Sci. Math. (Szeged)* **59** (1994), 603–612.